TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

MA 6351

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FOURIER SERIES

Question 1

Find the sum of the Fourier series of $f(x) = x + x^2$ in $-\pi < x < \pi$ at $x = \pi$.

Solution 1

$x = \pi$ is a discontinuous point in this interval . Therefore

$$f(x) = \frac{1}{2}[- \pi + (-\pi)^2 + \pi + \pi^2]$$

$$= \pi^2$$

Question 2

What is known as harmonic analysis?

Solution 2

The process of finding the Harmonics in the Fourier expansion of a function numerically is known as Harmonic analysis.

Question 3

State the Dirichlet's conditions for the existence of Fourier series of $f(x)$ in $[0,2\pi]$ with period 2π.

Solution 3

A function f(x) defined in (c, c+2l) can be expanded as an infinite trigonometric series of the form

$$f(x)= \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} +\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} ,$$

provided

1. f(x) is single valued and finite in (c,c+2l).
2. f(x) is continuous or piece-wise continuous with finite number of finite discontinuities in (c,c+2l).
3. f (x) has no or finite number of maxima or minima in (c,c+2l) are satisfied,then

the above 3 conditions are called the Dirichlet's conditions.

Question 4

let f(x) be defined in $(0,2\pi)$ by

$$f(x) = \begin{cases} \dfrac{1+\cos x}{\pi - x}, & 0 < x < \pi \\ \cos x, & \pi < x < 2\pi \end{cases}$$

and $f(x+2\pi) = f(x)$ with period 2π.

Solution 4

$$\lim_{x\to\pi-} f(x) = \lim_{x\to\pi-} \frac{-\sin x}{-1} \text{ (using l hospital's rule)}$$

$$= 0$$

$$\lim_{x\to\pi+} f(x) = \cos\pi$$

$$= -1$$

$$f(x) = \frac{1}{2}[f(\pi-) + f(\pi+)] \qquad \text{(average)}$$

$$= \frac{0+(-1)}{2} = -\frac{1}{2}$$

Question 5

If $f(x) = \begin{cases} cosx, & if\ 0 < x < \pi \\ 50, & if\ \pi < x < 2\pi \end{cases}$

and f(x) = f(x+2π) for all x, find the sum of the Fourier series of f(x) at x = π.

Solution 5

x= π is a discontinuous point.

Sum of the Fourier series of the function f(x) at x= π is,

$$f(\pi) = \frac{f(\pi-)+f(\pi+)}{2}$$

$$= \frac{\cos\pi+50}{2}$$

$$= \frac{-1+50}{2}$$

$$f(\pi) = \frac{49}{2}$$

Question 6

Find the coefficient b_5 of cos5x in the Fourier cosine series of the function $f(x)=\sin 5x$ in the interval $(0,2\pi)$

Solution 6

$$\text{since } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_5 = \frac{2}{\pi}\int_0^{2\pi} \sin 5x \cos 5x\,dx$$

$$= \frac{2}{\pi}\int_0^{2\pi} \frac{\sin 10x}{2}\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin 10x\,dx$$

$$= \frac{1}{\pi}\left[-\cos 10x/10\right]_0^{2\pi}$$

$$= 0$$

Question 7

Define Fourier cosine series in $[0,2l]$

Solution 7

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2l}$$

$$\text{where } a_0=\frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n=\frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{2l}\right)dx$$

Question 8

State Parseval's formula

Solution 8

The formula that expresses $\bar{y}^2$ in terms of a_0, a_n , b_n is known as parseval's formula.

That is , in Fourier series in (c,c+2*l*),

Then

$$\bar{y}^2 = \frac{1}{2l}\int_c^{c+2l}[f(x)]^2 dx$$

$$= \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2+b_n{}^2)$$

Question 9

Find the sine series for $f(x)=k$ in $(0,\pi)$

Solution 9

W.k.t the half range sine series of f(x) is in (0, l) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin nx$$

To find b_n:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{\pi}\int_0^{\pi} k \sin nx \, dx$$

$$= \frac{2k}{\pi}\int_0^{\pi} \sin nx \, dx \qquad = \frac{2k}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{2k}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right] \qquad = \frac{2k}{n\pi}\left[1 - \cos n\pi\right]$$

$$b_n = \begin{cases} 0 & \text{when n is even} \\ \frac{4k}{n\pi} & \text{when n is odd} \end{cases}$$

Question 10

State Euler's formula for Fourier coefficient of a function defined in (c,c+2l)

Solution 10

Fourier series in (c,c+2*l*) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$a_0=\frac{1}{l}\int_{c}^{c+2l} f(x)dx$$

$$a_n=\frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n=\frac{1}{l}\int_{c}^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$$

Question 11

Find the value of b_n for x^2+1 in $-1 \leq x \leq 1$.

Solution 11

Since $f(x) = x^2+1$ is an even function.

Therefore $b_n = 0$

Question 12

Define Root Mean square value of a function f(x) in $(0,2l)$

Solution 12

If a function y=f(x) is defined in (0,2*l*)

then $\sqrt{\frac{1}{2l}\int_0^{2l} y^2\,dx}$ is called the

root mean square value of y

and is denoted by $\bar{y}$.

Thus $\bar{y}^2 = \frac{1}{2l}\int_0^{2l} y^2\,dx$

Question 13

Find the RMS Value of $f(x)=x^2$ in $(-1,1)$

Solution 13

$$\text{RMS} = \sqrt{\frac{1}{2}\int_{-1}^{1} x^4\, dx}$$

$$= \sqrt{\frac{1}{2}\left(\frac{x^5}{5}\right)_{-1}^{1}}$$

$$= \sqrt{\frac{1}{2}\left(\frac{1}{5}+\frac{1}{5}\right)} = \sqrt{\frac{1}{2}\cdot\frac{2}{5}} = \frac{1}{\sqrt{5}}$$

Question 14

If $f(x) = 2x$ in the $(0,4)$ then find the value of a_2 in the Fourier series expansion.

Solution 14

Let f(x)=2x

W.k.t $a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$

since c=0, c+2*l*=4, *l*=2

Put n=2, $a_2 = \frac{1}{2}\int_0^4 2x\cos\frac{2\pi x}{2}\,dx$

$$= \int_0^4 x\cos\pi x\,dx$$

$$= \left[x\left(\frac{\sin\pi x}{\pi}\right) - \left(\frac{-\cos\pi x}{\pi^2}\right)\right]_0^4$$

$$= 4(0) + \frac{1}{\pi^2} - 0 - \frac{1}{\pi^2} = 0$$

Question 15

If $f(x)=x^3$, $-\pi < x < \pi$, find the constant term of its Fourier series.

Or

Obtain the first term of the Fourier series for the function $f(x) = x^3$, $-\pi < x < \pi$.

Solution 15

Since $f(x) = x^3$ is an odd function.

Therefore $a_0 = 0$

Question 16

Determine the value of b_n in the Fourier expansion of $x\sin x$ in $(-\pi , \pi)$

Solution 16

since xsinx is an even function in $(-\pi , \pi)$. Therefore

$$b_n = 0.$$

Question 17

Find b_n in the expansion of x^2 as a Fourier series in $(-\pi , \pi)$.

Solution 17

since $f(x)=x^2$ is an even function in $(-\pi , \pi)$.Therefore

$$b_n = 0.$$

Question 18

State Parseval's identity for the half range cosine expansion of f(x) in $(0,l)$.

Solution 18

$$2\int_0^l f(x)^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty} a_n^2$$

Where, $a_0 = 2\int_0^l f(x)\,dx$

$$a_n = \int_0^l f(x)\cos nx\,dx$$

Question 19

Obtain the sum of Fourier series for

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases} \text{ at } x=1$$

Solution 19

Here when x =1

$$f(1) = 1 \text{ in } 0 < x < 1$$

and

$$f(1) = 2 \text{ in } 1 < x < 2$$

Therefore x=1 is a point of discontinuity,

$$f(x) = \frac{f(1+0)+f(1-0)}{2}$$

$$= \frac{2+1}{2} = \frac{3}{2}$$

Question 20

Find the constant term in the expansion of $\cos^2 x$ as a Fourier series in the interval $(-\pi , \pi)$.

Solution 20

Since $\cos^2 x = \dfrac{1+\cos 2x}{2}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x + \frac{\sin 2x}{2}\right]_{-\pi}^{\pi}$$

$$=1$$

Question 21

Write a_0 , a_n in the expansion of $x+x^3$ as Fourier series in $(-\pi , \pi)$.

Solution 21

since $x+x^3$ is an odd function.

Therefore $a_0 = a_n = 0$.

Question 22

Define periodic function?

Solution 22

A function $f(x)$ is said to be a periodic function if for all x, $f(x+T) = f(x)$ where T is a positive constant. The least value of $T > 0$ is called the period of $f(x)$.

Question 23

Define complex form of Fourier series

Solution 23

The complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{\frac{in\pi x}{l}}$$

Where

$$c_n = \frac{1}{2l} \int_{c}^{c+2l} f(x) \, e^{\frac{-in\pi x}{l}} \, dx.$$

Fourier series will be updated then and there……….