TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

MA 6351

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FOURIER TRANSFORMS

Question 1

Write the Fourier transforms pair.

Solution 1

The Fourier transform of f(x) is defined by,

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

The inverse Fourier transform of F(s) is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx} ds$$

The Fourier transform F(s) of f(x) and the inverse Fourier transform $f(x) = F^{-1}\{F(s)\}$ are jointly called Fourier transform pair

Question 2

Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$

Solution 2

The Fourier sine transform of f(x) is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right] \quad \left[\text{since } \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}\right]$$

Question 3

Find the Fourier Cosine transform of e^{-ax} , $x > 0$.

Solution 3

The Fourier Cosine transform of f(x) is,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cossx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} cossx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \quad \left[\text{since } \int_0^\infty e^{-ax} cosbx \, dx = \frac{a}{a^2+b^2}\right]$$

Question 4

If F(s) is the Fourier transform of f(x), show that F[f(x-a)] = e^{ias} F(s)

Solution 4

W.K.T , $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

$\Rightarrow F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx}\, dx$

Put x-a = y , dx = dy

When x = $-\infty$, y = $-\infty$

When x = ∞, y = ∞

$\therefore F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{is(y+a)}\, dy$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isy} f(y) e^{ias}\, dy$

$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isy} f(y)\, dy$

$F[f(x-a)] = e^{ias} F(s)$

Question 5

Find Fourier cosine transform of e^{-2x}

Solution 5

W.k.t $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cossx\, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} cossx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2}{s^2+4}\right]$$

[since $\int_0^\infty e^{-ax} cossx\, dx = \frac{a}{s^2+a^2}$]

Question 6

If F(s) is the Fourier transforms of f(x),
show that the Fourier transform of $e^{iax}f(x)$ is F(s+a)

Solution 6

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x}\, dx \,|$$

$$F[e^{iax} f(x)] = F(s+a)$$

Question 7

State Parseval's identity for Fourier transform

Solution 7

If f(x) is a given function defined in (-∞, ∞), then it satisfies the identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

Where F(s) is the Fourier transform of f(x).

Question 8

State Fourier Integral Theorem.

Solution 8

If $f(x)$ is piecewise continuous, has piecewise continuous derivatives in every finite interval in $(-\infty,\infty)$ and absolutely integrable in $(-\infty,\infty)$, then

$$\mathbf{f(x)} = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos[s(x-t)]\,dt\,ds$$

Question 9

If F[f(x)] = F(s), then find $F[f(x)\cos ax]$

Solution 9

F[f(x)cosax] $= \frac{1}{2} F\left[f(x)\left(e^{ias} + e^{-ias}\right)\right]$

$= \frac{1}{2}\left[F\left(f(x)e^{ias}\right) + F\left(f(x)e^{-ias}\right)\right]$

[Since If $F[f(x)] = \bar{f}(s)$***, then*** $F\left[e^{-ias} f(x)\right] = \bar{f}$**(s+a)]**

$= \frac{1}{2}\bar{f}(s+a) + \frac{1}{2}\bar{f}(s-a)$

F[f(x)cosax] $= \frac{1}{2}\left[\bar{f}(s+a) + \bar{f}(s-a)\right]$

Question10

State inversion theorem for a complex Fourier transform.

Solution 10

If $F(s)$ denote the complex Fourier transform of $f(x)$, then the inverse Fourier transform is

$$\mathbf{f(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[s]\, e^{-isx}\, ds$$

Question 11

If F(s) is the Fourier transform of f(x), find the Fourier transform of f(ax) where a > 0.

Solution 11

W.k.t $\quad F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{-isx}\, dx$$

Put ax = y, adx = dy, $\quad dx = \frac{dy}{a}$

When x = -∞, y = -∞

When x = ∞ , y = ∞

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{\frac{-isy}{a}} \frac{dy}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\left(\frac{s}{a}\right)y}\, dy$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

Question 12

State the convolution theorem for Fourier transform.

Solution 12

The Fourier transform of the convolution of two functions is the product of their Fourier transforms .

(ie) if $\boldsymbol{F}[f(x)]$ =F(s) and $\boldsymbol{F}[g(x)]$ =G(s) , then

$$\boldsymbol{F}[f(x) * g(x)] = \mathbf{F}(\mathbf{s}).\mathbf{G}(\mathbf{s})$$

Question 13

Find the Fourier transform of

$$f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x \le a \text{ and } x > b \end{cases}$$

Solution 13

W.k.t $\quad F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{ikx} e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{i(k-s)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k-s)x}}{i(k-s)}\right]_{a}^{b}$$

$$= \frac{1}{i(k-s)\sqrt{2\pi}} \left[e^{i(k-s)b} - e^{i(k-s)a}\right]$$

Question 14

Find the Fourier sine transform of 1/x

Solution 14

W.k.t $\quad F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) sinsx\, dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{1}{x}\, sinsx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$$

[since $\int_0^{\infty} \frac{sinax}{x}\, dx = \frac{\pi}{2}$, a > 0]

$$= \sqrt{\frac{\pi}{2}}$$

Question 15

Write down the Fourier cosine transform pair formulae.

Solution 15

$$\mathbf{F_c[f(x)]} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$\mathbf{F^{-1}[\,F_c[f(x)]]} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c\,(f(x)) \cos sx \, dx$$

Question 16

If F_c (s) be the Fourier cosine transform of f(x), then prove that

$$F_c\,[f(x)\cos ax] = \frac{1}{2}\,[F_c(s+a) + F_c(s-a)\,]$$

Solution 16

We know that $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cossx \, dx$

$F_c[f(x) \, cosax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cosax \, cossx \, dx$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x). 2cosax \, cossx \, dx$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x). \{cos(s+a)x + cos(s-a)x\} \, dx$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, cos(s+a)x \, dx + \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, cos(s-a)x \, dx$

$F_c[f(x) \, cosax] = \frac{1}{2} [F_c(s+a) + F_c(s-a)]$

Question17

State the Fourier transforms of the derivatives of a function.

Solution 17

$$\mathrm{F}\left[\frac{d^n(f(x))}{dx^n}\right] = (-is)^n \mathrm{F(s)}$$

Question18

Find the Fourier sine transform of e^{-x}.

Solution 18

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) sinsx\, dx$$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} sinsx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+1}$$

$$\left[\text{since } \int_0^{\infty} e^{-ax} sinbx\, dx = \frac{b}{a^2 + b^2}\right]$$

Question 19

Find the Fourier cosine transform of

$$f(x) = \begin{cases} x \ , & 0 < x < \pi \\ 0 \ , & x > \pi \end{cases}$$

Solution 19

We know that $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^\pi x\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[x\left(\frac{\sin sx}{s}\right) - 1\left(\frac{-\cos sx}{s^2}\right)\right]_0^\pi$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \frac{\cos s\pi}{s^2} - \frac{1}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\cos s\pi - 1}{s^2}\right]$$

Question 20

State the shifting properties on Fourier transform

Solution 20

(i) If $F[f(x)] = F(s)$,then

$$F[f(x\text{-}a)] = e^{-ias}F(s)$$

(ii) If $F[f(x)] = F(s)$, then

$$F\left[e^{-iax}f(x)\right] = F(s+a)$$

Fourier Transform will be
updated
then and there……