TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

MA 6351

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Anna University

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Dept. of Mathematics. 9128-SAEC,Ramnad.

Z – TRANSFORM

AND

DIFFERENCE EQUATIONS

Question 1

Find the Z-transform of $\frac{1}{n!}$

Solution 1

$$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \ldots$$

$$= 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots\ldots$$

$$= e^{\frac{1}{z}}$$

[since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots$]

$$\therefore Z\left(\frac{1}{n!}\right) = e^{\frac{1}{z}}$$

Question 2

If $Z[f(n)] = f(z)$, what is $Z[f(n-k)]$?

Solution 2

W.k.t Z[f(n)] = $\sum_{n=0}^{\infty} f(n)$ z^{-n}

There fore

Z[f(n-k)] = $\sum_{n=0}^{\infty} f(n-k)$ z^{-n}

put m = n-k, n= m+k

$$= \sum_{m=-k}^{\infty} f(m) z^{-(m+k)}$$

$$= z^{-k} \sum_{m=0}^{\infty} f(m) z^{-m}$$

(since f(n) is causal sequence (ie) if f(n)=0,n<0)

$$= z^{-k} \bar{f}(z),$$

where $\bar{f}(z)$ = Z(f(n))

Question 3

Find Z[n]

Solution 3

$$Z[n] = \sum_{n=0}^{\infty} n\, z^{-n} \quad = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right]$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)^2}$$

(since $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots$)

$$= \frac{1}{z}\frac{z^2}{(z-1)^2} \qquad \mathbf{Z[n]} = \frac{z}{(z-1)^2}$$

Question 4

Form the difference equation from

$$y_n = a + b\,3^n$$

Solution 4

Given $y_n = a + b\,3^n$ -------------------- (1)

$y_{n+1} = a + b\,3^{n+1} = a+3b\,(3)^n$ ------------ (2)

$y_{n+2} = a + b\,3^{n+2} = a+9b\,(3)^n$ -----(3)

Eliminating a and b from (1), (2) and (3) we get,

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$y_n\,[9-3] - 1[9y_{n+1} - 3\,y_{n+2}] + 1\,[y_{n+1} - y_{n+2}] = 0$

$\Rightarrow \quad 6y_n - 9\,y_{n+1} + 3\,y_{n+2} + y_{n+1} - y_{n+2} = 0$

$\Rightarrow \quad 6y_n - 8\,y_{n+1} + 2\,y_{n+2} = 0$

$\Rightarrow y_{n+2} - 4\,y_{n+1} + 3\,y_n = 0$

Question 5

Find the value of $z[f(n)]$ when $f(n) = n\,a^n$.

Solution 5

$Z[n\,a^n] = Z[a^n.n]$

$= \{Z(n)\}_{z \to \frac{z}{a}}$

W.k.t $Z[n] = \dfrac{z}{(z-1)^2}$

$Z[n\,a^n] = \left\{\dfrac{z}{(z-1)^2}\right\}_{z \to \frac{z}{a}}$

$$= \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{z}{a} \cdot \frac{a^2}{(z-a)^2}$$

$$= \frac{az}{(z-a)^2}$$

Question 6

Find Z$\left[\frac{a^n}{n!}\right]$ in Z transform

Solution 6

$$Z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$$

$$= 1 + \frac{1}{1!}(az^{-1}) + \frac{1}{2!}(az^{-1})^2 + \ldots\ldots$$

$$= e^{az^{-1}}$$

$$\left[\text{ since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$$

$$\therefore Z\left[\frac{a^n}{n!}\right] = e^{\frac{a}{z}}$$

Question 7

Find $Z[e^{-iat}]$ using Z transform

Solution 7

Z[e^{-iat}] = $\sum_{n=0}^{\propto} e^{-ia(nT)} z^{-n}$

= $\sum_{n=0}^{\propto} (e^{-iaT})^n z^{-n}$

Since Z[a^n] = $\sum_{n=0}^{\propto} a^n z^{-n} = \frac{z}{z-a}$

Z[e^{-iat}] = $\frac{z}{z-e^{-iaT}}$

Question 8

State and prove initial value theorem in Z transform

Solution 8

If Z[f(n)] = F(z), then $\lim_{z\to\infty} F(z)$= f(0) = $\lim_{n\to 0} f(n)$

W.k.t Z[f(n)] = $\sum_{n=0}^{\infty} f(n)\, z^{-n}$

$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots\ldots\ldots\ldots\ldots\ldots$

ie) $F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots\ldots\ldots\ldots\ldots$

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots\ldots\ldots\ldots\ldots\right]$$

$= f(0)$ since $\lim_{z\to\infty} \frac{1}{z} = 0$

$\therefore \lim_{z\to\infty} F(z) = \lim_{n\to 0} f(n)$

Question 9

Find the

Z transform of

(n+1) (n+2)

Solution 9

$$Z[(n+1)(n+2)] = Z[n^2 + 3n + 2]$$

$$= Z[n^2] + 3Z[n] + 2Z[1]$$

$$Z[(n+1)(n+2)] = \frac{z(z+1)}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{z-1}$$

Question 10

Define Z transform of the sequence {f(n)}

Solution 10

Consider a sequence {f(n)}: {f(0) , f(1), f(2), …..} which is defined for all positive integers n = 0,1,2,……∞ .

Then Z transform of {f(n)} Is defined as

$Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$.

Question 11

Form a difference equation by eliminating arbitrary constants from

$$u_n = a\,2^{n+1}$$

Solution 11

Given $u_n = a\ 2^{n+1}$

$u_{n+1} = a\ 2^{n+2}$

$= a.2^{n+1}.2$

$= u_n.2$

$\mathbf{u_{n+1} = 2\ u_n}$

Question 12

State convolution theorem of

Z transform

Solution 12

(i) $Z[f(n) * g(n)] = F(z).G(z)$

where $Z[f(n)] = F(z)$ and

$Z[g(n)] = G(z)$

(ii) $Z[f(t) * g(t)] = F(z).G(z)$

where $Z[f(t)] = F(z)$ and

$Z[g(t)] = G(z)$

Question 13

Find the Z transform of $(-1)^n$

Solution 13

$$Z\{(-1)^n\} = \sum_{n=0}^{\infty} (-1)^n \, z^{-n}$$

$$= 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\ldots$$

$$= \left(1 + \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{1+\frac{1}{z}}$$

$$= \frac{z}{z+1}$$

Question 14

Find the Z transform of n^2

Solution 14

$$Z[n^2] = Z(n*n)$$

$$= -z \frac{d}{dz}[Z(n)]$$

since $Z(n) = \frac{z}{(z-1)^2}$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z \left[\frac{(z-1)^2.1 - z.2.(z-1)}{(z-1)^4}\right]$$

$$= -z \left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= -z \left[\frac{-z-1}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

Question 15

State final value theorem on Z transform

Solution 15

If Z[f(t)] = F(z), then

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)\,F(z)$$

Question 16

Find the Z-transform of $\left(\frac{-1}{3}\right)^n$

Solution 16

$$Z\left[\left(\frac{-1}{3}\right)^{n}\right] = \frac{z}{z-\left(\frac{-1}{3}\right)}$$

$$\left[\text{Since } Z(a^n) = \frac{z}{z-a}\right]$$

$$= \frac{z}{z+\frac{1}{3}}$$

$$= \frac{3z}{3z+1}$$

Question 17

State Initial value theorem in Z transform

Solution 17

If Z[f(n)] = F(z), then

$$\lim_{z \to \infty} F(z) = f(0) = \lim_{n \to 0} f(n)$$

Question 18

Find the inverse Z-transform of $\frac{z}{(z-1)(z-2)}$

Solution 18

Consider $\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$ ------

---------- (1)

$\Rightarrow z = A(z-2) + B(z-1)$

Put z = 1, -A = 1, A = -1

Put z = 2, B = 2.

∴ **(1) becomes**

$$\frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{2}{z-2}$$

$$z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$$

$$= z^{-1}\left[\frac{-1}{z-1}\right] + 2z^{-1}\left[\frac{2}{z-2}\right]$$

$$= (-1)^{n} + 2^{n}$$

$$= 2^{n} - 1$$

Question 19

Fin d z$[e^{-an}]$

Solution 19

$$z[e^{-an}] = z\left[\left(e^{-a}\right)^{n}\right]$$

$$\left[\text{ since } z[a^{n}] = \frac{z}{z-a}\right]$$

$$= \frac{z}{z-e^{-a}}$$

Question 20

Find the Z transform of $\sin \frac{n\pi}{2}$

Solution 20

w.k.t $z[\sin n\theta] = \dfrac{z\,sin\theta}{z^2 - 2zcos\theta + 1}$

put $\theta = \dfrac{\pi}{2}$,

$$z[\sin \frac{n\pi}{2}] = \frac{z\,sin\frac{\pi}{2}}{z^2 - 2zcos\frac{\pi}{2} + 1}$$

$$= \frac{z}{z^2 + 1}$$

Z-Transform will be updated then and there…….